Random Probabilistic Circuits (Supplementary material)

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A AN EXAMPLE OF DETERMINISTIC AND STRUCTURED DECOMPOSABLE XPC

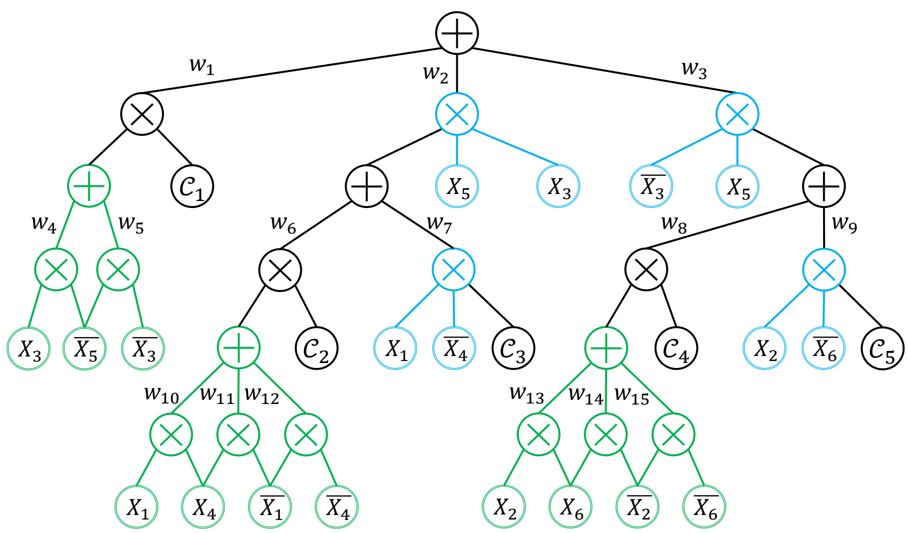

Figure 2: A deterministic and non-SD XPC learned setting $l = 2$ and $k = 3$ over RVs $\mathbf{X} = \{X_i\}_{i=1}^{7}$. The sub-circuits in blue (resp., green) encode $\mathcal{Q}$-regions (resp., $\mathcal{R}$-regions). Every CLT (represented as a multivariate node $\mathcal{C}_i$) is learned by following the traditional Chow-Liu tree algorithm [Chow and Liu, 1968].

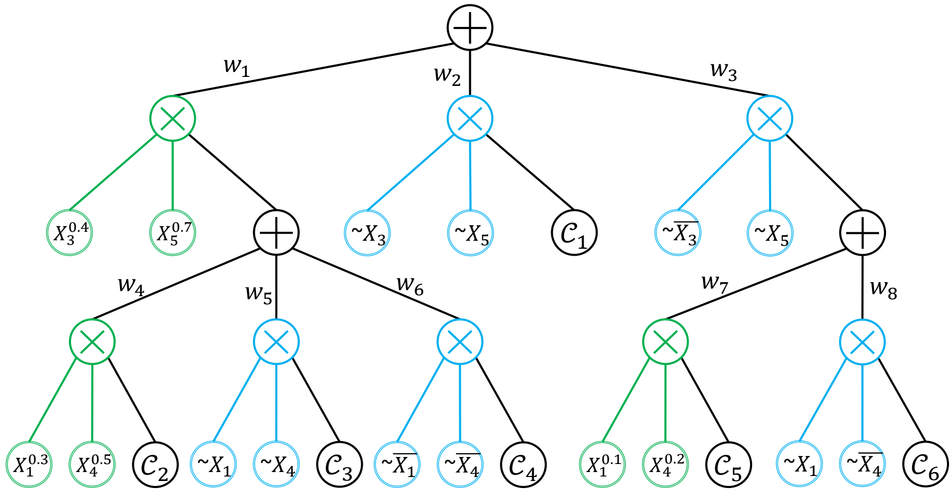

Figure 3: A non-deterministic and SD XPC learned setting $l = 2$ and $k = 3$ over RVs $\mathbf{X} = \{X_i\}_{i=1}^{7}$. The sub-circuits in blue (resp., green) encode $\mathcal{Q}$-regions (resp., $\mathcal{R}$-regions). $\sim X_i$ denotes the smoothed indicator of X_i and X_i^p denotes a Bernoulli node of X_i with parameter p. The CLTs (represented as multivariate nodes $\mathcal{C}_i$) have the same dependency tree—are normalized for the same vtree.

B PROOF. OF THEOREM 1.

We recall from [Chow and Liu, 1968] that the best first-order dependency tree $\widehat{\mathcal{T}}$ over a set of RVs $\mathbf{X} = \{X_i\}_{i=1}^{d}$ is:

$$\widehat{\mathcal{T}} = \arg\max_{\mathcal{T}} \sum_{i=1}^{d} \mathsf{MI}(X_i, X_{\tau(i)}), \tag{1}$$

where $X_{\tau(i)}$ is the parent of X_i in $\mathcal{T}$. Let $\{p_i\}_{k=1}^{n}$ be probability distributions on $\mathbf{X}$ and $\{q_i\}_{k=1}^{n}$ their corresponding approximations based on a same first-order dependency tree $\mathcal{T}$. We want to find $\widehat{\mathcal{T}}$ such that:

$$\widehat{\mathcal{T}} = \arg\min_{\mathcal{T}} \sum_{k=1}^{n} \mathbb{KL}(p_k || q_k) = \arg\max_{\mathcal{T}} \sum_{k=1}^{n} \mathbb{E}_{\mathbf{x}\sim p_k}[\log q_k(\mathbf{x})], \tag{2}$$

where $\mathbb{KL}$ denotes the Kullback-Leibler divergence. Thus considering a dependency tree $\mathcal{T}$:

$$\sum_{k=1}^{n} \mathbb{E}_{\mathbf{x}\sim p_k}[\log q_k(\mathbf{x})] = \sum_{k=1}^{n} \mathbb{E}_{\mathbf{x}\sim p_k}\left[\sum_{i=1}^{d} \log p_k(X_i | X_{\tau(i)})\right] \tag{3}$$

$$= \sum_{k=1}^{n}\left(\sum_{i=1}^{d} \mathsf{MI}_k(X_i; X_{\tau(i)}) + \mathbb{E}_{\mathbf{x}\sim p_k}\left[\sum_{i=1}^{d} \log p_k(X_i)\right]\right), \tag{4}$$

where $\mathsf{MI}_k(X, Y)$ is the mutual information of X and Y on p_k. Therefore:

$$\widehat{\mathcal{T}} = \arg\min_{\mathcal{T}} \sum_{k=1}^{n} \mathbb{KL}(p_k || q_k) = \arg\max_{\mathcal{T}} \sum_{k=1}^{n}\sum_{i=1}^{d} \mathsf{MI}_k(X_i; X_{\tau(i)}). \tag{5}$$

C TWENTY DENSITY ESTIMATION DATASETS

We report in Table 3 the adopted 20 real-world density estimation datasets [Haaren and Davis, 2012] and their statistics, already used to evaluate different tractable density estimators.

Table 3: Standard benchmark datasets and their statistics.

DATASET	RVS	TRAIN	VALID	TEST
NLTCS	16	16181	2157	3236
MSNBC	17	291326	38843	58265
KDD	64	180092	19907	34955
PLANTS	69	17412	2321	3482
AUDIO	100	15000	2000	3000
JESTER	100	9000	1000	4116
NETFLIX	100	15000	2000	3000
ACCIDENTS	111	12758	1700	2551
RETAIL	135	22041	2938	4408
PUMSB-STAR	163	12262	1635	2452
DNA	180	1600	400	1186
KOSAREK	190	33375	4450	6675
MSWEB	294	29441	3270	5000
BOOK	500	8700	1159	1739
EACHMOVIE	500	4525	1002	591
WEBKB	839	2803	558	838
ROUTERS-52	889	6532	1028	1540
20NEWS-GRP	910	11293	3764	3764
BBC	1058	1670	225	330
AD	1556	2461	327	491

D SINGLE MODELS

D.1 DENSITY ESTIMATION PERFORMANCES

We compare all the variants of XPCs. The average test-set log-likelihood is reported in Table 4

Table 4: Means and standard deviations of the best average test-set log-likelihoods over 40 runs for each variant of XPCs type in the space formed by $l = \{1, 2, 3\}$, $k = \{2, 3, 4, 8\}$ and $\delta = \{16, 32, 64, 128, 256, 512\}$. For every dataset, each row corresponds (from top to down) to XPC, XPC_{Det}, XPC^{SD} and $\text{XPC}^{\text{SD}}_{\text{Det}}$. For each model (row) we report the best value in bold.

	$l = 1$		$l = 2$			$l = 3$		
	$k = 2$	$k = 2$	$k = 3$	$k = 4$	$k = 2$	$k = 3$	$k = 4$	$k = 8$
NLTCS	**-6.10±0.03**	-7.03±0.36	-6.35±0.23	-6.14±0.02	-7.75±0.57	-6.92±0.41	-6.55±0.33	-6.15±0.02
	-6.10±0.03	-6.76±0.23	-6.26±0.13	-6.11±0.02	-7.16±0.31	-6.59±0.21	-6.37±0.17	-6.12±0.02
	-6.09±0.00	-7.03±0.58	-6.42±0.26	-6.12±0.01	-7.57±0.84	-6.69±0.35	-6.36±0.06	-6.14±0.02
	-6.09±0.00	-6.70±0.31	-6.27±0.13	-6.10±0.00	-6.77±0.31	-6.40±0.13	-6.27±0.03	-6.10±0.01
MSNBC	**-6.18±0.04**	-6.68±0.06	-6.38±0.07	-6.21±0.03	-6.78±0.06	-6.70±0.07	-6.54±0.10	-6.22±0.03
	-6.18±0.04	-6.51±0.06	-6.29±0.06	-6.19±0.03	-6.62±0.05	-6.49±0.08	-6.36±0.08	-6.19±0.03
	-6.21±0.02	-6.72±0.08	-6.45±0.10	-6.25±0.02	-6.75±0.07	-6.73±0.09	-6.62±0.08	-6.26±0.04
	-6.21±0.02	-6.50±0.06	-6.33±0.06	-6.23±0.02	-6.57±0.04	-6.49±0.06	-6.40±0.06	-6.24±0.04

KDD	**-2.21±0.01**	-2.23±0.02	**-2.21±0.01**	**-2.21±0.00**	-2.24±0.02	-2.22±0.01	**-2.21±0.01**	**-2.21±0.01**
	-2.22±0.01	-2.23±0.02	-2.22±0.01	-2.22±0.01	-2.23±0.02	**-2.21±0.01**	**-2.21±0.01**	-2.22±0.01
	-2.20±0.00	-2.24±0.02	-2.21±0.01	**-2.19±0.00**	-2.28±0.04	-2.24±0.03	-2.21±0.01	-2.20±0.00
	-2.20±0.00	-2.24±0.02	-2.21±0.01	**-2.20±0.00**	-2.26±0.03	-2.22±0.02	**-2.20±0.01**	**-2.20±0.00**
PLANTS	-13.83±0.15	-15.15±0.63	-14.02±0.26	-13.67±0.17	-16.50±0.90	-14.98±0.60	-14.33±0.49	**-13.56±0.09**
	-14.11±0.14	-15.26±0.34	-14.26±0.21	-13.97±0.09	-16.22±0.71	-14.94±0.45	-14.41±0.26	**-13.96±0.13**
	-14.93±0.00	-15.55±0.59	-14.75±0.25	**-14.55±0.04**	-15.55±0.81	-14.91±0.31	-14.67±0.07	-14.52±0.05
	-14.93±0.01	-15.37±0.47	-14.82±0.16	-14.67±0.03	-15.25±0.34	-14.86±0.15	-14.73±0.05	**-14.59±0.05**
AUDIO	-42.54±0.10	-42.36±0.24	-42.30±0.21	-42.38±0.12	-42.98±0.44	-42.33±0.29	**-42.20±0.20**	-42.32±0.13
	-42.63±0.11	-43.05±0.24	-42.78±0.13	-42.67±0.15	-43.44±0.35	-42.97±0.22	-42.85±0.25	-42.65±0.14
	-41.96±0.00	-42.31±0.47	-42.13±0.16	-42.08±0.03	-43.07±0.71	-42.58±0.38	-42.22±0.26	-42.11±0.03
	-41.97±0.00	-42.67±0.44	-42.29±0.16	-42.16±0.03	-43.00±0.51	-42.73±0.31	-42.49±0.24	-42.25±0.03
JESTER	-55.94±0.21	-55.52±0.34	-55.62±0.23	-55.73±0.14	-56.26±0.49	-55.65±0.34	**-55.47±0.29**	-55.61±0.21
	-56.09±0.22	-56.59±0.38	-56.25±0.25	-56.06±0.15	-56.85±0.45	-56.37±0.37	-56.19±0.33	**-56.00±0.23**
	-54.89±0.00	-55.15±0.52	-54.86±0.22	-54.84±0.02	-56.11±0.69	-55.20±0.32	-54.95±0.33	**-54.67±0.00**
	-54.94±0.00	-55.72±0.45	-55.16±0.23	-55.01±0.02	-56.12±0.52	-55.59±0.29	-55.42±0.31	-54.93±0.01
NETFLIX	-59.28±0.10	-59.23±0.20	-59.22±0.13	**-59.21±0.13**	-60.08±0.31	-59.44±0.19	-59.22±0.17	-59.22±0.13
	-59.29±0.10	-59.79±0.17	-59.43±0.15	**-59.28±0.13**	-60.26±0.24	-59.82±0.14	-59.57±0.16	-59.33±0.12
	-58.72±0.00	-59.17±0.51	-58.88±0.15	-58.79±0.02	-60.29±0.72	-59.27±0.27	-58.99±0.31	-58.87±0.01
	-58.73±0.00	-59.37±0.36	-58.99±0.17	-58.84±0.01	-59.85±0.41	-59.34±0.20	-59.17±0.21	-58.97±0.01
ACCIDENTS	-31.90±0.27	-32.76±0.38	-32.03±0.34	**-31.84±0.24**	-33.83±0.63	-32.77±0.40	-32.31±0.34	-31.92±0.27
	-31.88±0.37	-32.76±0.36	-32.09±0.34	-31.91±0.25	-33.77±0.56	-32.72±0.35	-32.32±0.33	-32.03±0.28
	-31.01±0.00	-32.02±0.37	-31.35±0.01	-31.35±0.01	-34.40±1.02	-32.06±0.28	-31.38±0.22	-31.17±0.21
	-31.03±0.00	-31.88±0.24	-31.35±0.01	-31.35±0.01	-32.78±0.44	-31.73±0.13	-31.30±0.15	-31.15±0.08
RETAIL	**-10.95±0.00**	-10.96±0.02	-10.96±0.02	-10.96±0.02	-11.01±0.04	-11.02±0.04	-11.02±0.03	-11.02±0.03
	-10.95±0.00	-10.96±0.01	-10.96±0.01	-10.96±0.01	-10.99±0.04	-11.00±0.03	-11.00±0.03	-11.00±0.03
	-10.97±0.00	-11.17±0.09	-11.04±0.07	-10.98±0.00	-11.24±0.09	-11.11±0.07	-11.06±0.03	-11.01±0.01
	-10.98±0.00	-11.06±0.09	-10.99±0.01	-10.99±0.00	-11.07±0.07	-11.02±0.05	-11.00±0.01	-11.00±0.01
PUMSB	-25.95±0.48	-27.32±0.76	-26.00±0.47	-25.69±0.37	-28.37±0.74	-27.03±0.65	-26.18±0.57	**-25.62±0.27**
	-26.14±0.52	-27.42±0.88	-26.18±0.46	**-25.90±0.41**	-28.29±0.68	-27.04±0.62	-26.27±0.58	-25.92±0.37
	-27.06±0.00	-27.19±0.06	-27.19±0.06	-27.19±0.06	-26.65±0.47	**-26.55±0.50**	**-26.55±0.50**	**-26.55±0.50**
	-27.06±0.00	-27.19±0.06	-27.19±0.06	-27.19±0.06	-26.66±0.48	**-26.56±0.51**	**-26.56±0.51**	**-26.56±0.51**
DNA	**-87.74±0.12**	-87.79±0.11	-87.79±0.11	-87.79±0.11	-87.92±0.11	-87.92±0.11	-87.92±0.11	-87.92±0.11
	-87.75±0.12	-87.79±0.08	-87.79±0.08	-87.79±0.08	-87.95±0.10	-87.95±0.10	-87.95±0.10	-87.95±0.10
	-87.46±0.00	-87.53±0.00	-87.53±0.00	-87.53±0.00	-87.54±0.00	-87.54±0.00	-87.54±0.00	-87.54±0.00
	-87.46±0.00	-87.53±0.00	-87.53±0.00	-87.53±0.00	-87.47±0.00	-87.47±0.00	-87.47±0.00	-87.47±0.00
KOSAREK	-11.25±0.07	-11.11±0.06	-11.09±0.09	-11.12±0.08	-11.08±0.08	**-11.03±0.05**	-11.04±0.06	-11.04±0.06
	-11.38±0.08	-11.35±0.10	-11.31±0.08	-11.34±0.08	-11.34±0.10	**-11.29±0.08**	**-11.29±0.09**	**-11.29±0.09**
	-10.97±0.00	-11.23±0.22	-11.05±0.16	**-10.92±0.02**	-11.40±0.15	-11.16±0.14	-11.04±0.06	-10.96±0.01
	-11.01±0.00	-11.25±0.14	-11.06±0.09	**-10.99±0.01**	-11.42±0.14	-11.20±0.14	-11.08±0.05	-11.03±0.00
MSWEB	**-10.19±0.02**	**-10.19±0.02**	**-10.19±0.02**	**-10.19±0.02**	**-10.19±0.02**	**-10.19±0.02**	**-10.19±0.02**	**-10.19±0.02**
	-10.19±0.01	**-10.19±0.02**	**-10.19±0.02**	**-10.19±0.02**	-10.20±0.02	-10.20±0.02	-10.20±0.03	-10.20±0.03
	-10.12±0.00	-10.43±0.13	-10.13±0.03	**-10.12±0.01**	-10.65±0.14	-10.44±0.13	-10.27±0.11	-10.13±0.02
	-10.12±0.00	-10.31±0.07	-10.15±0.02	-10.15±0.00	-10.38±0.06	-10.29±0.06	-10.20±0.05	-10.13±0.01
BOOK	-37.66±0.32	-37.14±0.32	-37.34±0.40	-37.34±0.40	-36.73±0.37	**-36.68±0.34**	**-36.68±0.34**	-36.68±0.34
	-37.79±0.19	-37.69±0.21	-37.69±0.21	-37.69±0.21	-37.51±0.29	**-37.50±0.21**	**-37.50±0.21**	**-37.50±0.21**
	-36.35±0.00	-36.00±0.17	-36.00±0.17	-36.00±0.17	**-35.90±0.00**	**-35.90±0.00**	**-35.90±0.00**	**-35.90±0.00**
	-36.97±0.00	-36.83±0.10	-36.83±0.10	-36.83±0.10	-36.60±0.00	**-36.60±0.00**	**-36.60±0.00**	**-36.60±0.00**

EMOVIE	-62.19±2.13	-60.61±1.69	-60.98±1.87	-61.38±1.99	-59.78±1.95	-59.64±1.92	-59.41±2.01	**-59.20±1.21**
	-63.91±1.24	-63.37±1.41	-63.46±1.31	-63.46±1.31	-62.86±1.54	**-62.62±1.58**	-62.72±1.30	-62.74±1.31
	-59.07±0.01	-58.44±0.54	-57.94±0.39	-57.80±0.15	-57.90±0.89	-58.06±0.20	**-57.71±0.57**	-57.98±0.00
	-60.56±0.00	-60.86±0.24	-60.49±0.06	-60.49±0.06	**-59.99±0.65**	-60.47±0.22	-60.35±0.00	-60.35±0.00
WEBKB	-163.31±0.35	-162.82±0.88	-162.88±0.88	-162.88±0.88	**-162.42±1.10**	-162.53±0.92	-162.44±1.00	-162.44±1.00
	-163.43±0.23	**-163.33±0.29**	**-163.33±0.29**	**-163.33±0.29**	-163.33±0.34	-163.33±0.34	-163.33±0.34	-163.33±0.34
	-159.56±0.00	-158.44±0.35	-159.11±0.41	-159.11±0.41	**-158.30±0.09**	-158.52±0.63	-158.48±0.00	-158.48±0.00
	-161.30±0.00	-161.19±0.22	-161.91±0.09	-161.91±0.09	**-160.93±0.29**	-161.26±0.00	-161.26±0.00	-161.26±0.00
ROUTERS	-94.05±0.69	-93.45±1.15	-93.69±1.13	-93.71±1.11	**-93.00±1.34**	-93.13±0.80	-93.12±0.84	-93.12±0.84
	-94.32±0.33	-94.32±0.51	**-94.29±0.37**	**-94.29±0.37**	-94.29±0.48	-94.29±0.48	-94.29±0.48	-94.29±0.48
	-88.13±0.00	-87.94±0.52	-87.95±0.45	-87.95±0.45	-88.89±0.80	-88.18±0.61	-87.91±0.52	**-87.91±0.29**
	-88.92±0.00	-89.71±0.58	-89.33±0.42	-89.33±0.42	-90.66±0.55	-90.58±0.65	-89.47±0.35	-89.43±0.18
20NEWS	-162.96±1.12	-161.58±0.77	-162.04±0.96	-162.04±0.96	**-160.73±0.56**	-161.06±0.67	-161.06±0.63	-161.06±0.63
	-164.26±0.61	**-163.81±0.53**	-163.95±0.44	-163.95±0.44	-163.86±0.84	-163.95±0.85	-164.03±0.87	-164.03±0.87
	-158.38±0.00	**-157.31±0.58**	-157.55±0.43	-157.44±0.16	-157.79±0.80	-157.86±0.41	-157.71±0.52	-158.16±0.01
	-159.58±0.00	-160.61±1.07	-159.73±0.46	**-159.37±0.17**	-161.12±0.65	-160.68±0.44	-160.41±0.42	-160.20±0.01
BBC	-261.88±0.34	**-261.81±0.28**	**-261.81±0.28**	**-261.81±0.28**	-261.96±0.48	-261.96±0.48	-261.96±0.48	-261.96±0.48
	-262.04±0.77	**-261.96±0.58**	**-261.96±0.58**	**-261.96±0.58**	-262.24±0.98	-262.24±0.98	-262.24±0.98	-262.24±0.98
	-262.33±0.00	-260.52±0.00	-260.52±0.00	-260.52±0.00	-259.45±1.26	**-258.79±0.00**	**-258.79±0.00**	**-258.79±0.00**
	-262.91±0.00	-261.41±0.00	-261.41±0.00	-261.41±0.00	**-260.62±0.00**	**-260.62±0.00**	**-260.62±0.00**	**-260.62±0.00**
AD	-16.40±0.04	**-16.38±0.11**	**-16.38±0.11**	**-16.38±0.11**	-16.40±0.12	-16.40±0.12	-16.40±0.12	-16.40±0.12
	-16.62±0.13	**-16.61±0.13**	**-16.61±0.13**	**-16.61±0.13**	-16.66±0.26	-16.66±0.26	-16.66±0.26	-16.66±0.26
	-16.02±0.00	-16.02±0.00	-16.02±0.00	-16.02±0.00	-16.26±0.10	-16.26±0.10	-16.05±0.20	**-15.97±0.08**
	-16.28±0.00	**-16.28±0.00**	**-16.28±0.00**	**-16.28±0.00**	-16.39±0.07	-16.39±0.07	-16.39±0.07	-16.39±0.07

D.2 TRAINING TIME AND CIRCUIT SIZES

The training time and the circuit size for our models are reported in Table 5.

Table 5: Each row reports δ-values/training times in seconds (top) and circuit sizes (bottom) obtained for learning for the best XPC models reported in Table 4.

DATASET	XPC	XPC_{Det}	XPC^{SD}	$\text{XPC}^{\text{SD}}_{\text{Det}}$
NLTCS	64 / 0.06 2765±458	64 / 0.08 2765±458	64 / 0.05 4335±67	64 / 0.05 4401±67
MSNBC	512 / 0.36 3575±546	512 / 0.71 3575±546	512 / 0.31 4887±125	512 / 0.31 4887±125
KDD	64 / 1.29 21331±1591	128 / 2.09 16089±1312	128 / 0.64 12951±109	128 / 1.02 13040±109
PLANTS	64 / 0.25 31403±1301	128 / 0.18 19138±2135	64 / 0.12 22242±532	128 / 0.1 13960±644
AUDIO	256 / 0.18 20897±1232	512 / 0.12 11396±740	128 / 0.17 29275±42	128 / 0.17 29317±42
JESTER	512 / 0.06 6530±722	256 / 0.11 12584±870	128 / 0.13 25351±69	128 / 0.11 20273±29
NETFLIX	256 / 0.17 20239±1655	512 / 0.12 12510±976	128 / 0.21 39811±57	128 / 0.21 39868±57
ACCIDENTS	256 / 0.15 16252±2742	256 / 0.09 8994±2417	128 / 0.09 11906±15	128 / 0.09 11921±15
RETAIL	512 / 0.04 1098±416	512 / 0.04 1098±416	512 / 0.09 6645±7	512 / 0.09 6651±7
PUMSB-STAR	128 / 0.45 43737±6494	128 / 0.35 32356±7266	128 / 0.1 8703±1442	128 / 0.12 8866±1480
DNA	256 / 0.04 3927±888	256 / 0.04 3927±888	512 / 0.02 2615±1	512 / 0.02 2616±1
KOSAREK	256 / 0.28 13730±3381	256 / 0.29 13617±2872	256 / 0.27 20898±34	256 / 0.3 20938±34
MSWEB	512 / 0.13 2298±714	512 / 0.13 2298±714	512 / 0.27 12130±5	512 / 0.27 12135±5
BOOK	256 / 0.37 11610±4205	256 / 0.35 11227±4306	256 / 0.39 13505±4	256 / 0.38 13678±28
EACHMOVIE	128 / 0.53 19913±8846	256 / 0.31 11375±5785	64 / 1.58 80294±9009	256 / 0.5 21369±3124
WEBKB	256 / 0.89 17026±6503	512 / 0.42 7719±3334	128 / 2.59 61915±13804	512 / 0.87 17122±2
ROUTERS-52	128 / 2.88 55112±18036	512 / 0.65 10479±5049	256 / 2.91 67594±2067	512 / 1.52 36440±5
20NEWS-GRP	512 / 2.11 37138±10983	512 / 2.2 38557±10433	512 / 3.28 65838±2880	512 / 3.34 65881±2884
BBC	512 / 0.49 7755±1930	512 / 0.51 7755±1932	128 / 1.86 36578±6	256 / 0.77 14578±1
AD	64 / 0.94 11661±2170	64 / 1.01 12368±3157	16 / 21.59 298553±10013	128 / 1.69 22093±20

E MIXTURE MODELS

E.1 EXPCS

The best average test-set log-likelihoods for every EXPC type are reported in Table 6.

Table 6: Best average test-set log-likelihoods for every EXPC type in the space formed by $l = \{1, 2, 3\}$, $k = \{2, 3, 4, 8\}$, $\delta = \{16, 32, 64, 128, 256, 512\}$ and $M = \{2, 5, 10, 15, 20, 25, 30, 40\}$. For every dataset, each row corresponds (from top to down) to: EXPC, EXPC_{Det}, EXPC^{SD} and $\text{EXPC}^{\text{SD}}_{\text{Det}}$.

	$l = 1$		$l = 2$			$l = 3$		
	$k = 2$	$k = 2$	$k = 3$	$k = 4$	$k = 2$	$k = 3$	$k = 4$	$k = 8$
NLTCS	-6.00	-6.30	-6.04	-6.00	-6.54	-6.20	-6.10	**-6.01**
	-6.00	-6.26	-6.04	-6.00	-6.48	-6.18	-6.09	-6.01
	-	-6.43	-6.09	**-6.05**	-6.50	-6.29	-6.15	-6.10
	-	-6.38	-6.08	**-6.05**	-6.42	-6.23	-6.11	-6.08
MSNBC	**-6.12**	-6.40	-6.19	-6.12	-6.56	-6.38	-6.25	-6.12
	-6.12	-6.35	-6.17	-6.12	-6.46	-6.33	-6.21	-6.12
	-	-6.43	-6.22	**-6.18**	-6.54	-6.44	-6.33	-6.19
	-	-6.38	-6.20	**-6.17**	-6.48	-6.38	-6.29	-6.19
KDD	-2.13	-2.14	-2.13	**-2.13**	-2.13	-2.13	-2.13	-2.13
	-2.13	-2.13	-2.13	**-2.14**	-2.13	-2.13	-2.13	-2.13
	-	-2.15	-2.17	-2.19	-2.17	**-2.15**	-2.16	-2.18
	-	-2.15	-2.17	-2.20	-2.17	**-2.16**	-2.16	-2.19
PLANTS	-12.41	-13.14	-12.53	-12.35	-13.91	-13.03	-12.59	**-12.24**
	-12.41	-13.11	-12.51	-12.35	-13.78	-12.96	-12.57	**-12.24**
	-	-14.73	**-14.19**	-14.55	-14.82	-14.45	-14.39	-14.46
	-	-14.61	**-14.21**	-14.62	-14.76	-14.46	-14.43	-14.50
AUDIO	-39.84	-40.03	-39.80	-39.78	-40.46	-40.06	-39.89	**-39.75**
	-39.84	-40.12	-39.86	-39.80	-40.53	-40.14	-39.97	**-39.77**
	-	-40.96	-40.96	-41.74	-41.66	**-40.91**	-40.96	-41.91
	-	-41.00	-41.05	-41.76	-41.65	**-40.97**	-41.07	-42.01
JESTER	-52.72	-52.89	-52.65	-52.67	-53.27	-52.89	-52.73	**-52.60**
	-52.72	-52.96	-52.75	-52.68	-53.35	-52.97	-52.84	**-52.63**
	-	**-53.43**	-53.52	-54.59	-54.01	-53.49	-53.41	-54.68
	-	**-53.51**	-53.68	-54.63	-54.03	-53.61	-53.59	-54.76
NETFLIX	-56.43	-56.79	-56.50	**-56.38**	-57.39	-56.77	-56.64	-56.41
	-56.43	-56.93	-56.58	**-56.40**	-57.41	-56.88	-56.75	-56.43
	-	-57.66	-57.68	-58.46	-58.18	-57.67	**-57.58**	-58.60
	-	-57.68	-57.80	-58.46	-58.14	-57.73	**-57.69**	-58.62
ACCIDENTS	-29.63	-30.63	-29.76	**-29.54**	-31.33	-30.35	-29.97	-29.55
	-29.63	-30.58	-29.74	-29.53	-31.28	-30.31	-29.93	**-29.54**
	-	-31.71	-31.36	-31.36	-33.12	-31.45	**-31.02**	-31.08
	-	-31.60	-31.36	-31.36	-31.99	-31.32	**-30.99**	-31.08
RETAIL	-10.83	-10.82	-10.82	-10.82	-10.81	-10.81	**-10.81**	-10.81
	-10.83	-10.81	-10.81	-10.81	-10.81	**-10.80**	-10.79	-10.79
	-	-10.94	-10.96	-10.97	-11.00	-10.94	**-10.94**	-10.96
	-	-10.91	-10.95	-10.98	**-10.90**	-10.91	-10.95	-10.97
PUMSB	-23.78	-24.87	-23.86	-23.58	-25.61	-24.56	-23.90	**-23.55**
	-23.78	-24.78	-23.83	-23.57	-25.49	-24.44	-23.85	**-23.53**
	-	-27.21	-27.21	-27.21	-26.16	**-26.06**	-26.06	-26.06
	-	-27.21	-27.21	-27.21	-26.16	**-26.05**	-26.05	-26.05
DNA	**-84.83**	-85.75	-85.47	-85.47	-86.25	-85.69	-85.62	-85.53
	-84.83	-85.78	-85.48	-85.48	-86.24	-85.73	-85.62	-85.54
	-	-87.25	-87.53	-87.53	-86.72	**-86.61**	-87.54	-87.54
	-	-86.85	-87.25	-87.25	**-85.09**	-86.30	-87.47	-87.47
KOSAREK	-10.68	-10.64	-10.66	-10.66	-10.65	-10.63	**-10.61**	-10.62
	-10.68	-10.66	-10.66	-10.66	-10.67	-10.64	**-10.63**	-10.63
	-	-10.87	**-10.77**	-10.89	-11.04	-10.85	-10.81	-10.96
	-	-10.86	**-10.81**	-10.95	-11.01	-10.87	-10.85	-11.03

MSWEB	-10.09	-9.94	-9.90	-9.90	-9.84	**-9.78**	-9.78	-9.77
	-10.09	-9.94	-9.90	-9.90	-9.84	**-9.78**	-9.78	-9.78
	-	-10.09	**-9.93**	-10.07	-10.24	-10.05	-9.97	-10.05
	-	-10.02	**-9.94**	-10.12	-10.06	-9.97	-9.93	-10.09
BOOK	-35.72	-35.16	-35.23	-35.23	**-34.85**	-35.02	-35.02	-35.02
	-35.63	-35.40	-35.40	-35.40	**-34.98**	-35.16	-35.16	-35.16
	-	-34.86	-35.37	-35.68	**-34.75**	-35.08	-35.41	-35.90
	-	-35.25	-36.28	-36.28	**-35.14**	-35.71	-36.60	-36.60
EMOVIE	-54.50	-53.78	-53.78	-53.87	-53.08	-53.23	-53.15	**-52.93**
	-54.65	-54.06	-54.13	-54.30	-53.35	-53.65	**-53.44**	-53.45
	-	-55.35	-56.05	-57.27	-55.23	**-54.82**	-55.42	-57.98
	-	-55.88	-57.39	-60.15	**-55.26**	-55.70	-56.47	-60.35
WEBKB	-153.97	-153.92	-155.16	-155.23	**-154.03**	-153.93	-154.17	-154.17
	-154.29	-154.36	-155.33	-155.49	**-154.42**	-154.49	-154.66	-154.66
	-	-153.62	-157.22	-157.22	**-153.67**	-154.17	-156.28	-157.89
	-	-154.45	-159.88	-159.88	**-154.23**	-155.58	-158.22	-161.26
ROUTERS	-85.61	-85.32	-85.71	-85.28	-85.07	**-84.82**	-85.31	-85.04
	-85.89	-85.96	-86.12	-85.72	-85.53	**-85.27**	-85.74	-85.65
	-	-85.19	-85.17	-86.48	-85.38	**-84.70**	-85.02	-86.94
	-	-85.51	-85.93	-87.25	-85.49	**-85.09**	-85.96	-88.61
20NEWS	-154.48	-153.98	-154.47	-154.35	-153.85	**-154.13**	-154.45	-154.48
	-154.60	-154.54	-154.80	**-154.71**	-154.64	-154.91	-155.09	-155.07
	-	-153.78	-155.10	-156.86	**-153.75**	-154.02	-154.52	-157.26
	-	-154.35	-156.26	-158.63	**-154.21**	-154.56	-155.49	-160.10
BBC	**-237.96**	-242.58	-243.68	-243.72	-250.91	-251.07	-251.33	-251.33
	-239.01	-247.14	-246.98	-247.43	-251.35	-251.64	-251.85	-251.85
	-	-248.43	-253.19	-260.52	**-248.34**	-250.16	-251.50	-258.79
	-	-249.46	-255.95	-261.41	**-248.79**	-252.21	-255.59	-260.62
AD	-15.90	-14.39	-14.23	-14.23	-13.95	**-13.61**	-13.60	-13.60
	-15.94	-14.28	-14.14	-14.14	-13.78	**-13.45**	-13.45	-13.45
	-	-16.03	-16.03	-16.03	-15.66	-15.59	**-15.50**	-15.67
	-	-16.31	-16.31	-16.31	**-15.59**	-15.73	-15.68	-15.87

E.2 COMPETITORS

The average test-set log-likelihoods for best EXPCs are reported in Table 7.

Table 7: Average test-set log-likelihoods for best EXPCs, SOTA ensembling techniques and single model competitors. In the first half of the table the best results for ensembles of bagged (CNet^{40}) and boosted (CNet$^{40}_{\text{boost}}$) entropy based CNets taken from [Rahman and Gogate, 2016], ensembles of bagged CNets learned with dCSN and ensemble of extremely randomized CNets as in [Di Mauro et al., 2017]. The remaining columns report the comparison to other models employing more sophisticated models such as ID-SPN [Rooshenas and Lowd, 2014], RAT-SPN [Peharz et al., 2019], and Learn-SPN [Gens and Domingos, 2013].

	ensembles					competitors		
DATASET	EXPC	XCNet^{40}	CNet^{40}	CNet$^{40}_{\text{boost}}$	dCSN^{40}	RAT	ID-SPN	Learn-SPN
NLTCS	**-6.00**	**-6.00**	**-6.00**	-6.01	**-6.00**	-6.01	-6.02	-6.11
MSNBC	-6.12	-6.06	-6.08	-6.15	-6.05	**-6.04**	**-6.04**	-6.11
KDD	**-2.12**	-2.13	-2.14	-2.15	-2.15	-2.13	-2.13	-2.18
PLANTS	-12.24	**-11.99**	-12.32	-12.67	-12.59	-13.44	-12.54	-12.99
AUDIO	**-39.75**	-39.77	-40.09	-39.84	-40.19	-39.96	-39.79	-40.50
JESTER	**-52.51**	-52.65	-52.88	-52.82	-52.99	-52.97	-52.86	-75.98
NETFLIX	-56.38	-56.38	-56.55	-56.44	-56.69	-56.85	**-56.36**	-57.33
ACCIDENTS	-29.54	-29.31	-29.88	-29.45	-29.27	-35.49	-26.98	-30.04
RETAIL	**-10.79**	-10.93	-10.84	-10.81	-11.17	-10.91	-10.85	-11.04
PUMSB-STAR	-23.53	-23.44	-23.98	-23.46	-23.78	-32.53	-22.41	-24.78
DNA	-84.83	-84.96	-81.07	-85.67	-85.95	-97.23	-81.21	-82.52
KOSAREK	-10.61	-10.72	-10.74	**-10.60**	-10.97	-10.89	**-10.60**	-10.99
MSWEB	-9.78	**-9.66**	-9.77	-9.74	-9.93	-10.12	-9.73	-10.25
BOOK	**-34.14**	-36.35	-35.55	-34.46	-37.38	-34.68	**-34.14**	-35.89
EACHMOVIE	-52.38	-51.72	-53.00	-51.53	-54.14	-53.63	**-51.51**	-52.49
WEBKB	-152.44	-153.01	-153.12	-152.53	-155.47	-157.53	**-151.84**	-158.21
ROUTERS-52	-84.70	-84.05	-83.71	-83.69	-86.19	-87.37	**-83.35**	-85.07
20NEWS-GRP	-152.26	-153.89	-156.09	-153.12	-156.46	-152.06	**-151.47**	-155.93
BBC	-237.96	-238.47	**-237.42**	-247.01	-248.84	-252.14	-248.93	-250.69
AD	**-13.45**	-14.20	-15.28	-14.36	-15.55	-48.47	-19.05	-19.73
AVG. RANK	**2.00**	2.25	3.25	2.85	4.25			
	2.75	3.15	4.3	3.8	5.85	6	**2.5**	6.9

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